

Explosive condensation in a mass transport model

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We study a far-from-equilibrium system of interacting particles, hopping between sites of a 1d lattice with a rate which increases with the number of particles at interacting sites. We find that clusters of particles, which initially spontaneously form in the system, begin to move at increasing speed as they gain particles. Ultimately, they produce a moving condensate which comprises a finite fraction of the mass in the system. We show that, in contrast with previously studied models of condensation, the relaxation time to steady state decreases as an inverse power of $\ln L$ with system size L and that condensation is instantaneous for $L \rightarrow \infty$.

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Recent studies in non-equilibrium statistical physics show that diverse phenomena such as jamming in traffic flow [1], polydisperse hard spheres [2], wealth condensation in macroeconomies [3], hub formation in complex networks [4], pathological phases in quantum gravity [5], and general problems of phase separation [6] can be understood by the condensation transition. Condensation occurs when the global density of a conserved quantity (mass, wealth etc.) exceeds a critical value, and manifests itself as a finite fraction of the total system mass localized in space. A well-studied, fundamental model is the Zero-range Process (ZRP) which may serve as either a microscopic or effective description of non-equilibrium condensation [6–9]. In this model particles hop to the right on a closed chain of L sites with rates $u(m) \cong 1 + \gamma/m$ depending only on the number of particles $m > 0$ at the departure site. The condensate, which exists in this model for $\gamma > 2$ when density of particles is above some critical value, remains static once it has formed, melting and reforming very rarely [9]. This is caused by attractive interactions between particles expressed in $u(m)$: the more particles are in the condensate, the slower it evolves.

In this work we demonstrate a novel mechanism of non-equilibrium condensation motivated by processes such as gravitational clustering [10], formation of droplets in clouds or on inclined surfaces due to collisions [11], and differential sedimentation [12], where aggregation of particles speeds up in time as a result of increasing exchange rate of particles between growing clusters. For example, raindrops falling through the mist increase their velocity when gaining mass, which causes them to accrete mass even faster. To better understand the difference between the dynamical nature of the condensate in such processes and the static condensation which has previously been studied [1–5, 13], we consider a microscopic model of particles hopping between sites of a 1d lattice with rate $u(m, n) \sim (mn)^\gamma$ which increases with the numbers m, n of particles at interacting sites. We shall show that for $\gamma > 2$ condensation occurs through a contrast-

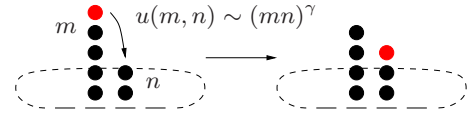


Figure 1: Model definition: a particle hops from site with m to site with n particles with rate $u(m, n)$.

ing dynamical mechanism to that previously considered — the formation of the condensate happens on a very fast time scale and we term it *explosive*. By considering the microscopic processes of the dynamics we show that the condensate moves with speed $v \sim L^\gamma$ which increases with system size L . We argue that each cluster of particles has a chance to develop into the condensate in finite time. It then follows from extreme value statistics that the time to form of the condensate *decreases* with system size as $(\ln L)^{1-\gamma}$, in contrast to ZRP. This counter-intuitive result means that condensation is instantaneous for $L \rightarrow \infty$.

Model definition: The model we consider comprises M particles hopping to the right between sites of a periodic chain of length L as in the ZRP. Although partial asymmetry may also be considered, we restrict ourselves here to the case of totally asymmetric hopping: a particle hops from site i to site $i + 1$ with rate $u(m_i, m_{i+1})$ where m_i, m_{i+1} are the occupancies of the departure and arrival sites, respectively. We assume the factorized form

$$u(m, n) = (v(m) - v(0))v(n), \quad (1)$$

where the function $v(m)$ grows as a power of m

$$v(m) = (\epsilon + m)^\gamma \sim m^\gamma, \quad (2)$$

with $\epsilon \ll 1$ and $\gamma > 0$ [19]. Equation (1) implies that $u(0, n) = 0$ and that for large m, n , $u(m, n) \sim (mn)^\gamma$ is the bigger the more particles are located on *both* sites. This has dramatic consequences for the dynamics. Comparing simulation results of this model to ZRP dynamics

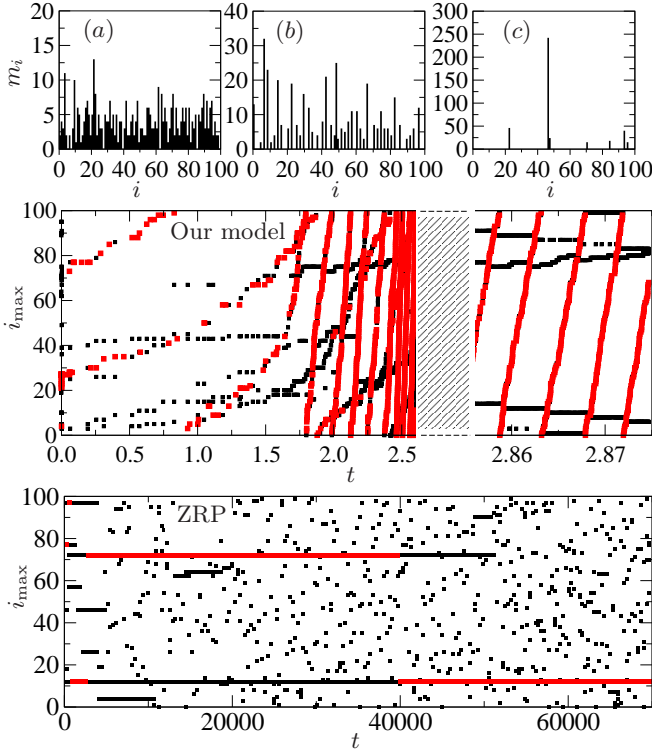


Figure 2: Top: state of the system $\{m_i\}$ for $v(m) = (m+0.1)^3$ and $L = 100, M = 400$ at different times: (a) initial condition with randomly distributed particles, (b) the rise of microscopic clusters of particles, separated by empty sites, (c) the macroscopic cluster (condensate) close to the steady state. Middle: positions of five most occupied sites (red squares for the largest cluster) as a function of time in this model. Bottom: the same plot for ZRP condensation for $u(m, n) = 1 + 3/m$ shows completely different dynamics.

in Fig. 2 reveals some striking differences (see also animations in Supp. Material [14]). In ZRP, initial microscopic clusters are first formed, but they coalesce and grow quickly, until two macroscopic clusters are left. These slowly merge into the final macroscopic condensate by exchanging particles through the other sites which form the fluid background. In our model, particles also aggregate into clusters (see Fig. 2b) but then these clusters start to move in the direction of hopping particles. This process speeds up in time; some clusters move faster as they gain particles in collisions, and one of them - the condensate - starts to dominate (Fig. 2c). Due to the rapid nature of this process we call it explosive condensation. The speed $v \equiv di_{\max}/dt$ at which the condensate travels through the system stabilizes after the system reaches the steady state. The motion of the condensate is similar to the “slinky”-like motion of a non-Markovian model [15]. Finally, smaller clusters move in the opposite direction to the main condensate at each collision.

The dynamics thus differs significantly from the zero-range process. Surprisingly, both models share simi-

lar static properties. In fact, they belong to a class of processes that have the important property that the steady state probability $P(\{m_i\})$ of a configuration with m_1, \dots, m_L particles at sites i, \dots, L factorizes:

$$P(\{m_i\}) = \prod_{i=1}^L f(m_i), \quad (3)$$

with $f(n)$ defined as

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^n \prod_{k=1}^n \frac{u(1, k-1)}{u(k, 0)}. \quad (4)$$

Equation (3) requires two conditions on $u(m, n)$ [14], which are satisfied for our model (1) and the ZRP (where $u(m, n) = 1 + \gamma/m$ for $m > 0$ and $u(0, n) = 0$). In both cases we can choose $f(1) = f(0)$ and obtain from Eq. (4) the large m behaviour $f(m) \sim m^{-\gamma}$. It is known [7] that for a power-law $f(m)$, condensation happens when the density of particles $\rho = M/L$ exceeds the critical density $\rho_c = \lim_{z \rightarrow 1} zF'(z)/F(z)$, where $F(z) = \sum_m f(m)z^m$ and z plays the role of fugacity. For $\gamma > 2$, ρ_c is finite but for $\gamma < 2$, $\rho_c \rightarrow \infty$. Therefore, condensation is possible only for $\gamma > 2$ and for $\rho > \rho_c$, which marks the transition between condensation/no condensation regimes [7].

We now come back to the dynamics of our process and investigate what determines the speed of clusters and the condensate, how the clusters collide, how long it takes to reach the steady state, and how this time depends on the initial condition. We are interested in the limit of large M, L and fixed density $\rho = M/L$. For our choice $v(m) = (\epsilon + m)^\gamma$ and $\gamma > 2$, we obtain $f(m) \cong f(0)\epsilon^\gamma m^{-\gamma}$ for $m > 0$ and the critical density, $\rho_c = \sum_m m f(m) / \sum_m f(m) \approx \epsilon^\gamma \zeta(\gamma - 1) \ll 1$ where $\zeta(\gamma - 1)$ is the Riemann zeta function. As the critical density is low one can make the simplifying approximation that the clusters move in an otherwise empty system.

Let us first calculate the speed at which the cluster of m particles moves through the system. We assume that at $t = 0$ the cluster occupies site i , so that $m_i = m$, and that there are no particles at sites $i - 1, i + 1$. The time τ it takes to move the cluster to site $i + 1$ is the sum of times t_m, t_{m-1}, \dots, t_1 it takes to move one particle to the right if the cluster has $m, m - 1, \dots, 1$ particles, respectively. Each n th hop is a random process with average duration t_n given by the inverse of the hopping rate $u(n, m - n)$, thus

$$\langle \tau \rangle = \sum_{n=1}^m \frac{1}{u(n, m - n)}. \quad (5)$$

Recalling that for condensation we are interested in $\gamma > 2$ and using Eqs. (1-2), we obtain that $\langle \tau \rangle \approx (\epsilon m)^{-\gamma}$, which shows that larger clusters move faster. The condensate, which has $\approx M$ particles, moves $1/\langle \tau \rangle \approx (\epsilon M)^\gamma = (\epsilon \rho)^\gamma L^\gamma$ sites per unit time, in agreement with simulations: for parameters from Fig. 2 we have measured

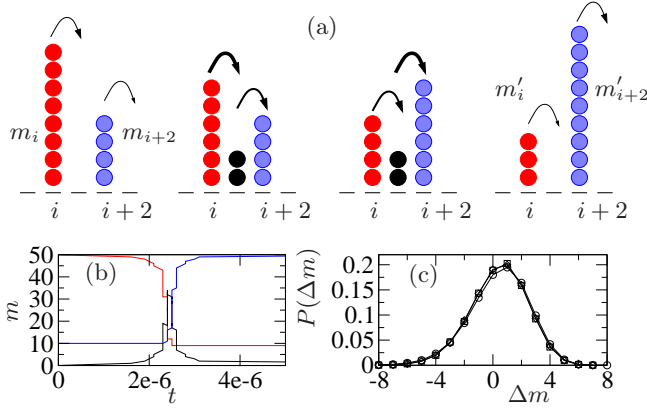


Figure 3: Top (a): A collision of two condensates having initially m_i and m_{i+2} particles (left) proceeds through exchange of particles at site $i+1$ (middle pictures). Arrows of different sizes show relative magnitudes of hopping rates. After the collision (right), the masses are m'_i and m'_{i+2} , with $\langle m'_i - m_i \rangle < 0$, $\langle m'_{i+2} - m_{i+2} \rangle > 0$. Bottom left (b): example of stochastic evolution of $m_i(t)$ (red), $m_{i+1}(t)$ (black), and $m_{i+2}(t)$ (blue). Bottom right (c): probability distribution $P(\Delta m)$ of the difference $\Delta m = m_i - m'_i$ for $m_i = 25, 50, 100$ (circles, squares, diamonds) and $m_{i+2} = 10, \epsilon = 0.1, \gamma = 3$. In all cases $\langle \Delta m \rangle \approx 0.4$.

the speed 63850 ± 100 whereas the formula $(\epsilon M)^\gamma$ gives 64000.

To understand what happens when two condensates collide with each other, we assume that a bigger condensate with m_i particles approaches a smaller one with m_{i+2} particles from the left, and that they are separated by an empty site $i+1$, see Fig. 3a. Initially, the dynamics is dominated by hops from site i to site $i+1$ because $u(m_i, m_{i+1}) \propto m_i^\gamma$ is bigger than $u(m_{i+1}, m_{i+2}) \propto m_{i+2}^\gamma$. As particles accumulate at site $i+1$, $u(m_{i+1}, m_{i+2})$ grows and $u(m_i, m_{i+1})$ decreases until they become comparable. This happens when $m_i \approx m_{i+2}$ since $u(m, n) \sim (mn)^\gamma$ is symmetric for large m, n . Then the second half of the process becomes a time-reversed and space-inverted version of itself (see Fig. 3b); one can think of the flow from i to $i+2$ as a flow from $i+2$ to i in reversed time which is identical to the flow from i to $i+2$ in the first half of the process. Due to this time-reversal symmetry, the final configuration becomes the initial configuration reflected around site $i+1$, modulo random fluctuations. In the case of (2), the slightly broken symmetry of $u(m, n)$ produces a small net current of particles from smaller to bigger clusters (see Fig. 3c).

We now venture to draw the following picture of condensation dynamics. First, small clusters are formed randomly from the initial state. For a system of size L there will be $N = O(L)$ such clusters. Subsequently, these clusters move ballistically between collisions, which are almost elastic. One of them soon collects more particles than the rest and starts moving at increasing speed, gaining mass and becoming the final condensate. Let

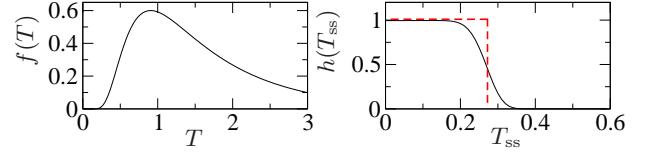


Figure 4: Sketches of $f(T)$ (left) and $h(T_{ss}) = \exp(-N \int_0^{T_{ss}} f(T) dT)$ (right). $h(T_{ss})$ can be approximated by a step function (see text).

us calculate the time T_{ss} for the system to relax to stationary state. Each cluster will go through a series of collisions and either dissolve into the background or become the condensate; in either case we can associate a relaxation time T to each cluster (with $T = \infty$ if the cluster disappears). Then T_{ss} will be the minimal time out of T_1, \dots, T_N relaxation times for all N clusters:

$$T_{ss} = \min \{T_1, \dots, T_N\}. \quad (6)$$

The relaxation process of a particular cluster of initial mass m_0 is a series of transitions at times t_n at which it moves by one site to the right and (possibly) exchanges a chunk of mass Δm_n with other clusters:

$$m_n = m_{n-1} + \Delta m_n, \quad (7)$$

$$t_n = t_{n-1} + \Delta t_n. \quad (8)$$

Here Δt_n is the time between two jumps and is exponentially distributed as

$$p_n(\Delta t_n) = \lambda_n e^{-\lambda_n \Delta t_n}, \quad (9)$$

where $\lambda_n \propto m_n^\gamma$ is the speed of the cluster. Let us calculate the probability distribution $f(T)$ of the relaxation time $T = \Delta t_1 + \Delta t_2 + \dots$. Numerical simulations suggest that the mass m_n increases linearly through the collisions. We may thus assume that $m_n \propto n$ and $\lambda_n \simeq An^\gamma$ for large n . Then T is a sum of independent exponential random variables and $f(T)$ is given by

$$f(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega T} \tilde{f}(\omega), \quad (10)$$

where $\tilde{f}(\omega)$ is the product of characteristic functions of exponential distributions (9):

$$\tilde{f}(\omega) = \prod_{n=1}^{\infty} \tilde{p}_n(\omega) = \prod_{n=1}^{\infty} \frac{1}{1 - i\omega/\lambda_n}. \quad (11)$$

We expect that $f(T)$ has the shape depicted in Fig. 4 and that it decays to zero for $T \rightarrow 0$. The large- ω behaviour of $\tilde{f}(\omega)$, which corresponds to small- T behaviour of $f(T)$, is given by [14]

$$\tilde{f}(\omega) \simeq -i(2\pi)^{\gamma/2} \sqrt{\frac{i\omega}{A}} \exp \left[-\frac{2\pi i(i\omega/A)^{1/\gamma}}{e^{2\pi i/\gamma} - 1} \right]. \quad (12)$$

Now, we must invert the Fourier transform to recover $f(T)$. For small T , this may be done by the saddle point approximation to the integral over ω (dominated by $\omega = O(T^{-\gamma/(\gamma-1)})$) and one obtains

$$f(T) \propto CT^{(1-3\gamma)/(2(\gamma-1))} \exp \left[-B(AT)^{-\frac{1}{\gamma-1}} \right], \quad (13)$$

where B, C are some real, positive constants. If we assume that each cluster evolves independently, the relaxation time (6) of the system becomes the minimum out of N independent random variables distributed according to $f(T)$. Extreme values statistics tells us that the distribution $P(T_{ss})$ is given by

$$P(T_{ss}) = Nf(T_{ss}) \left[\int_{T_{ss}}^{\infty} f(T) dT \right]^{N-1}, \quad (14)$$

and integrating by parts and expanding for $f(T)$ small,

$$\langle T_{ss} \rangle \cong \int_0^{\infty} \exp \left(-N \int_0^{T_{ss}} f(T) dT \right) dT_{ss}. \quad (15)$$

Knowing the small- T behaviour (13) of $f(T)$, we can calculate the average (15) for large N as follows. The function $h(T_{ss}) = \exp(-N \int_0^{T_{ss}} f(T) dT)$ approaches a step function for large N , see Fig. 4. The integral (15) over T_{ss} , then becomes $\langle T_{ss} \rangle \cong \int_0^{\infty} h(T_{ss}) dT_{ss} \cong t_0$, where t_0 is the position of the step in h , which can be identified as the point at which $h'' = 0$, yielding

$$f'(t_0) = Nf^2(t_0). \quad (16)$$

Inserting the short-time behaviour (13) of f into this condition one obtains

$$CN(\gamma - 1) = B \exp \left[B(At_0)^{-\frac{1}{\gamma-1}} \right] A^{-\frac{1}{\gamma-1}} t_0^{-\frac{\gamma}{\gamma-1}}. \quad (17)$$

Taking logarithms yields

$$t_0 \simeq \left(\frac{1}{\beta} \left[\ln N - \ln \left(\frac{\beta}{\gamma - 1} \right) - \frac{1}{2} \ln t_0 \right] \right)^{1-\gamma}. \quad (18)$$

Thus, recalling $N = O(L)$ and $\langle T_{ss} \rangle \cong t_0$, the relaxation time asymptotically decreases as

$$\langle T_{ss} \rangle = c_2(c_3 + \ln L)^{1-\gamma}. \quad (19)$$

This form crosses over from $\langle T_{ss} \rangle = 1/(\text{const} + O(\ln L))$ for small L to $\langle T_{ss} \rangle = O((\ln L)^{1-\gamma})$ for large L . This differs much from ZRP-like models where $\langle T_{ss} \rangle \sim L^2$ grows with L [7, 9]. Since the time to steady state decreases with L , an infinite system relaxes instantaneously. This is reminiscent of instantaneous gelation known from the theory of coagulation processes [17]. In fact, our model provides a non-trivial example of instantaneous gelation in a spatially-extended system. However, the model and its effective description in terms of colliding clusters differ from coagulation processes in that there is exchange

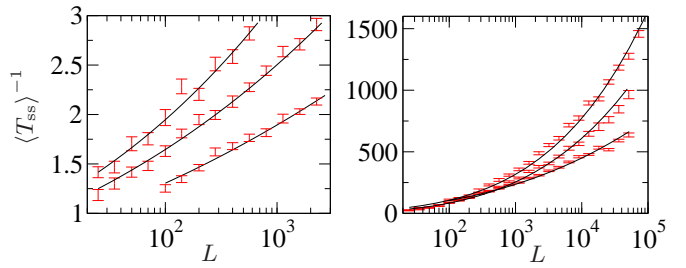


Figure 5: $\langle T_{ss} \rangle^{-1}$ obtained in numerical simulations (points) and from formula (19) fitted to data points (lines). In all cases the density $\rho = 2$ and $\gamma = 3, 4, 5$ (curves from bottom to top). Left: $v(m) = (0.3 + m)^\gamma$, every 5th site has initially 10 particles. Right: $v(m) = (1 + m)^\gamma$ particles are distributed randomly in the initial state. $\langle T_{ss} \rangle^{-1}$ for different γ differ by orders of magnitude and hence they have been rescaled to plot them together.

of particles between clusters rather than coagulation (a model with exchange of particles has been studied in Ref. [18], see Supp. Material for more details).

In the above derivation we assumed that λ_n is strictly proportional to n^γ , and that the proportionality coefficient is the same for all clusters. This is valid only if all clusters have the same initial size $m = 1$. To account for fluctuations of cluster sizes one should take the product (11) not from $n = 1$ but from some $n_0 > 0$, with n_0 changing from cluster to cluster. However, this does not modify the asymptotic behaviour of $\tilde{f}(\omega)$, it only increases the constant c_3 in Eq. (19). We have checked numerically evaluating Eqs. (11), (10) and (15) that $\langle T_{ss} \rangle$ for $n_0 > 1$ has much stronger finite-size corrections and behaves as $\sim 1/(\text{const} + O(\ln L))$ for a wide range of L . Although c_3 may in principle be calculated from our theory for $n_0 > 1$, in practice it is simplest to treat c_3 as a free parameter. In this way Eq. (19) fits numerical simulations very well. To check this, we measured the time it took the biggest cluster to reach the mean steady-state size of the condensate, $M - L\rho_c$. In Fig. 5 we compare Eq. (19) with $\langle T_{ss} \rangle$ obtained in simulations, for different initial conditions. We plot $\langle T_{ss} \rangle^{-1}$ because it shows convincingly that $\langle T_{ss} \rangle^{-1}$ grows to infinity for $L \rightarrow \infty$, and therefore $\langle T_{ss} \rangle \rightarrow 0$ in this limit.

In conclusion, we have elucidated a form of dynamic condensation which happens in far-from-equilibrium system of hopping particles. In contrast to previously studied models, the condensate moves through the system and its dynamics speeds up in time—hence we term the condensation “explosive”. The relaxation is dominated by the process of initial coalescence which is the slowest stage of condensate formation, at variance with previously studied models of condensation such as the ZRP where this stage is the fastest. It remains to be seen whether condensation can be made “explosive” also in models which do not have a factorized steady state, such as those with spatially extended condensates [13].

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